Reference Point Based NSGA-III for Preferred Solutions

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***Abstract*—The recent advances in evolutionary many-objective optimization (EMOs) have allowed for efficient ways of finding a number of diverse trade-off solutions in three to 15-objective problems. However, there are at least two reasons why the users are, in some occasions, interested in finding a part, instead of the entire Pareto-optimal front. First, after analyzing the obtained trade-off solutions by an EMO algorithm, the user may be interested in concentrating in a specific preferred region of the Pareto-optimal front, either to obtain additional solutions in the region of interest or to investigate the nature of solutions in the preferred region. Second, the user may already have a well-articulated preference among objectives and is straightaway interested in finding preferred solutions. In this paper, we suggest a reference point based evolutionary many-objective optimization procedure for achieving both of these purposes. Additionally, we suggest an extended version of a previously proposed reference-point based evolutionary multi-objective optimization method. Our proposed procedures are capable of handling more than one reference point simultaneously. We demonstrate the working of our proposed procedures on a number of test and real-world problems. The results are encouraging and suggest the use of the concept to other evolutionary many-objective optimization algorithms for further study.**

***Index Terms*—Reference point approach, interactive multi-objective decision making, multi-objective optimization, EMO.**

I. INTRODUCTION

After developing efficient algorithms for solving two and three-objective optimization problems [1]–[5], evolutionary multi-objective optimization (EMO) researchers have suggested new algorithms for handling four and more objectives [6]–[8]. These evolutionary many-objective optimization (EMnO[1]) algorithms are designed to find multiple trade-off solutions with the help of a set of guided reference directions. The basic idea of these methods is to search for points in parallel along each reference direction and to be as close to the ideal point as possible. The whole idea behind finding multiple Pareto-optimal solutions is that the nature of trade-off among solutions can be obtained first before choosing a preferred solution for implementation. However, with the increase in objective dimensionality, the number of points needed to represent the true nature of the Pareto-optimal front increases exponentially. Getting an idea of the shape of the Pareto-optimal front is one aspect but analyzing a large number of solutions to finally choose a single preferred solution becomes cumbersome and almost impossible for decision-makers (DMs) in a practical application. Thus, it is expected that DMs will have to settle with a series of optimization and decision-making tasks to effectively solve many-objective optimization problems.

If some preference information is available before performing any optimization task, DMs are, in effect, interested in finding only a small part of the Pareto-optimal front that corresponds to their desired preference properties. In this case, instead of finding the entire Pareto-optimal front, DMs would expect an EMO algorithm to directly focus its population members in their preferred region of interest. The existing EMnO algorithms have not been extended for this focused search for a part of the Pareto-optimal front. However, there exists a few past studies on EMO algorithms, which were combined with multi-criterion decision-making (MCDM) ideas for this purpose.

When absolutely no preference information is available before optimization, DMs are expected to follow a two-step procedure in which EMnO algorithms should be applied first to find a representative set of Pareto-optimal points to its entirety and then analyze them to finally focus on one or more regions of interest. While the first task is clearly a many-objective optimization process, the second task can also be achieved with another type of optimization task. The latter optimization task involves finding a number of focused points on one or more specific parts of the Pareto-optimal region. This latter optimization task is the main focus of this study in the context of many-objective optimization. The task can also be used for another purpose. If the former optimization task through an EMnO algorithm exhibits a hole, a gap, or finds only a few solutions in the regions of interest, the latter optimization can be applied to reveal the true nature of the Pareto-optimal front in these regions of interest.

In this paper, we extend the idea of reference point

[1]EMnO term is introduced here to differentiate from EMO, which is used for multi-objective optimization.

based NSGA-II, developed mainly for solving two and three-objective optimization problems [9], and propose a reference point based NSGA-III (or R-NSGA-III) for solving higher objective problems. Like in R-NSGA-II, we allow multiple reference points to be considered simultaneously. Since the R-NSGA-III method is expected to focus on a part of the Pareto-optimal front, it is likely to be faster than the original NSGA-III procedure. The working principle of R-NSGA-III is demonstrated by solving a number of many-objective optimization test problems and real-world problems. The performance has also been compared with the existing R-NSGA-II procedure and with a more balanced version of it. We attempt to highlight both aspects of the decision-making tasks discussed in the above paragraph: (i) find more trade-off points in the regions of interest, and (ii) validate if there exists or does not exist any Pareto-optimal point in regions where a former EMnO optimization run has failed to find any solution.

In the remainder of the paper, we discuss the motivation for finding preferred Pareto-optimal solutions in Section II. Then, we provide a brief description of a previously-proposed R-NSGA-II procedure in Section III. A balanced version of R-NSGA-II procedure to efficiently solve many-objective optimization problems is presented in Section IV. Thereafter, we make a brief introduction to the NSGA-III procedure in Section V and then present our proposed R-NSGA-III procedure in detail in Section VI. Results using these methods are presented in Section VII. Finally, conclusions of the study and a number of future study ideas are presented in Section VIII.

II. Motivation for Finding Preferred Pareto-optimal Solutions

Most EMO studies have concentrated in finding a representative set of the entire Pareto-optimal front, however, in practice the task of finding a number of trade-off solutions cannot be the last act. Somehow, a single preferred solution must be arrived at so that that solution can be implemented in practice. This decision-making task of choosing a single preferred solution should not be dismissed as a trivial task, as the whole multi-criterion decision-making field, started in early seventies, is still a very much active field for research and application. We believe that efficient computational algorithms that are directed by preference information from DMs can be developed to help choose a single preferred solution in two ways:

1) Select one of the optimized Pareto solutions by preference information provided by DMs, and
2) Take the optimized Pareto solutions as baseline starting points and then search for new and additional trade-off points in the regions of interest.

Both the above approaches constitute a two-step procedure of the complete optimization-cum-decision-making procedure. While the first approach is relatively easier, here, we focus on the second approach, which involves an additional optimization procedure.

There is another motivation for finding focused solutions. If the preference-based optimization procedure can be made computationally efficient, it can be used to analyze and validate different parts of the trade-off frontier obtained by the EMO procedure. For example, if the first EMO application has discovered a hole, a gap, or a few points in certain preferred part of the trade-off frontier, the preference based approach can focus there and investigate if the initial findings were really true. Such a computational procedure will provide more confidence about various regions of interest to the DM, before choosing a preferred solution.

III. Existing EMO Methods for Preferred Solutions

EMO algorithms for finding a part of the Pareto-optimal front were suggested before. The use of cone domination [10], [11] was suggested to find a subset of Pareto-optimal solutions, but the approach can only be applied to problems having convex Pareto-optimal front. Branke and Deb [11] suggested an approach with a projected distance based niching concept, but it is also not generic enough to be applied to find any arbitrary part of the Pareto-optimal front. Certain ideas with modified domination principle were proposed [12], but they too are not flexible and generically applicable. All these methods have not been applied to many-objective problems yet to test their suitability.

Instead of making an elaborate description of different preference based EMO methods, here, we discuss two methods which are more generically applicable and are related to our proposed R-NSGA-III procedure. These methods suggested the use of reference points (or aspiration points) by the decision-makers [13]–[15]. It is important to note that aspiration points can be specified anywhere on the objective space, as desired by the DM. The goal in these studies was to find Pareto-optimal points which are close to the supplied aspiration points.

A. *R-NSGA-II*

First, we provide a brief description of the R-NSGA-II procedure, suggested in 2006 [**?**], for finding a preferred part of the Pareto-optimal front. The preference information was provided by one or more reference points (or aspiration points) by following a multi-criterion decision-making (MCDM) approach originally proposed by Wierzbicki [13] in 1980. The R-NSGA-II procedure needed an ϵ-parameter indicating the minimum distance between two neighboring solutions in the objective space desired among the final solutions. If a large ϵ value is chosen, a large extent of solutions near each reference point can be obtained, and vice versa.

The R-NSGA-II procedure extends the working of NSGA-II procedure as follows. Instead of using a crowding distance based niching selection, as done in NSGA-II, a *clearing* based niching was adopted. For every non-dominated front starting with the first (the best) front, population members are sorted according increasing normalized Euclidean distance from every given aspiration point. Then, the closest member to each aspiration point is assigned the same rank of one. Thereafter, all population members which are within ϵ distance away from rank one members are *cleared* (meaning that they are temporarily not considered). The next closest member to each aspiration point is then assigned a rank of two. Thereafter, members within ϵ from rank two members are cleared. This process is continued until all front members are either assigned a rank or is cleared. The procedure is then repeated for the second front members and so on. After all fronts are considered, the above procedure is then repeated with cleared population members of the first front and subsequent ranks are assigned. After repeating for second, third, and so on fronts, the ranking is done for the remaining cleared points. This procedure is continued until all population members are assigned a rank. Finally, a binary tournament selection is performed on the merged population (parent and offspring) with the assigned ranks and half the merged population is chosen for the next generation.

The original study implemented the concept and successful results on two and three-objective problems were presented. The effect of the ϵ parameter was also demonstrated. The idea also worked on one five-objective and one 10-objective test problems due to the focused approach despite dealing with more than three objectives.

IV. Proposed Balanced R-NSGA-II or BR-NSGA-II

The original R-NSGA-II applied the niching operation independently to each aspiration point. In some problems, the process may end up finding unequal number of points for each aspiration point. We propose a balanced R-NSGA-II (BR-NSGA-II) approach which makes a better balance of solutions for each aspiration point.

In BR-NSGA-II method, for the last front which could not be selected as a whole, solutions closer to each aspiration point is chosen one at a time by maintaining a distance of ϵ as before, but this time the number of points to be chosen for each aspiration point depends on the number of solutions that have been already chosen from the previously accepted better fronts for the aspiration point. The process is balanced. In the case of two aspiration points ($\mathbf{z}^{(1)}$ and $\mathbf{z}^{(2)}$), let us say that all previously accepted fronts together have $n_1 = 40$ and $n_2 = 35$ points associated with the two aspiration points, respectively. From the last front, we need to choose remaining $m = 25$ points (assuming a population of size 100). Then, we choose $m_1 = 10$ closest points from $\mathbf{z}^{(1)}$, but ϵ distant from each other, and $m_2 = 15$ closest points from $\mathbf{z}^{(2)}$, but ϵ distant from each other. This allows exactly 50% population members allocated for each of the two aspiration points. Due to this balanced approach, there is no additional selection operator used. As another example, if $n_1 = 60$ and $n_2 = 30$, then all remaining $m = 10$ points will be selected from the second list of the final front members, thereby making the final number is representative solutions as 60 and 40 for two aspiration points, respectively, thereby making a better balance of population members for both aspiration points. In the event of the first non-dominated front being the final front (which happens after a few generations and quite early for many-objective optimization problems), there is no earlier solution ($n_1 = n_2 = 0$) assigned to aspiration points and an effort is made to choose equal number of points for each aspiration point. It is interesting to note that when a single aspiration point is supplied, both original R-NSGA-II and the above BR-NSGA-II are quite the same, except the stochasticity effect of the binary tournament selection operator used in R-NSGA-II. Thus, in BR-NSGA-II, the niching operation needs to be applied only to the last front members and also there is no need for any selection operator in BR-NSGA-II algorithm.

V. NSGA-III for Many-Objective Optimization

In 2014, Deb and Jain [7] proposed NSGA-III procedure which was able to rectify NSGA-II's inability to extend to more than three objectives by introducing a guidance through a number of pre-defined reference directions. The procedure is briefly described here, as the proposed reference point based NSGA-III method is based on this approach.

NSGA-III is exactly the same as NSGA-II until the merged parent and offspring population is applied with its selection operator. First, a set of H well-distributed reference points are chosen on a unit hyperplane using Das and Dennis's method [16]. Each reference point is joined with the origin to establish H reference directions in the positive objective coordinate space. All members of the merged population, at every generation, are *normalized* using a systematic extreme point update strategy mainly by using population-minimum and population-maximum objective values. Each member is then *associated* with a particular reference direction using the orthogonal distance of a member to a reference direction. Thereafter, a *niching* methodology is used to choose a diverse set of solutions by providing equal emphasis to each reference direction. No additional selection operator was needed, as the population size was kept almost the same as the number of reference directions. NSGA-III algorithm was tested on three to 15-objective optimization problems and in every case it was able to find a well-converged and well-distributed set of near-Pareto-optimal solutions on constrained [8] and unconstrained [7] problems. For constraints, a constraint-domination based selection operator [1] was introduced.

VI. Proposed R-NSGA-III Method

R-NSGA-III extends the NSGA-III procedure [7] by introducing a new reference point generation method according to user-supplied aspiration points, while using the same genetic operators and survival selection process. Say, K aspiration points are supplied by the user in M-dimensional objective space (where M is the number of objectives):

$$\mathbf{r}^{(k)} = \left(z_1^{(k)}, z_2^{(k)}, \ldots, z_M^{(k)}\right), \quad k = 1, 2, \ldots, K. \qquad (1)$$

Each aspiration point is first normalized using NSGA-IIIs normalization procedure. Knowing $\left(f_i^{\min}, f_i^{\max}\right)$ for each objective supplied by the normalization code, we obtain normalized aspiration points as follows:

$$\bar{\mathbf{r}}^{(k)} = \left(\frac{z_1^k - f_1^{\min}}{f_1^{\max} - f_1^{\min}}, \ \ldots \ , \ \frac{z_M^k - f_M^{\min}}{f_M^{\max} - f_M^{\min}}\right), \quad k = 1 \ \ldots \ K. \qquad (2)$$

We then compute the intercepts of the unit hyperplane and the vectors from the ideal point to each normalized reference point.

$$\dot{\mathbf{r}}^{(k)} = \frac{(\mathbf{p}_0 - \mathbf{l}_0)}{\bar{\mathbf{r}}^{(k)} \cdot \hat{n}} \cdot \bar{\mathbf{r}}^{(k)} + \mathbf{l}_0, \quad k = 1, 2, \ldots, K, \tag{3}$$

where $\hat{n} = (1, 1, \ldots, 1)^T/\sqrt{M}$, representing the normal vector for the hyperplane and $\mathbf{p}_0 = (1, 0, \ldots, 0)^T$ (one of the extreme points on the plane). We define $\mathbf{l}_0$ as the ideal point $(0, 0, \ldots, 0)^T$ representing a point on the line. The new point $\dot{\mathbf{r}}^k$ is a point that lies on the unit hyperplane. Next, $H = \binom{M+p-1}{p}$ Das and Dennis's points ($\mathbf{h}^{(j)}$) are created on a unit hyperplane (using a suitable gap p). These points are then shrunk, using a factor, μ, as follows:

$$\bar{\mathbf{h}}^{(j)} = \mu \mathbf{h}^{(j)}, \quad \mu \in (0, 1). \tag{4}$$

The shrunken Das-Dennis points are then shifted to the unit hyperplane by using a vector formed using the centroid ($\mathbf{g}$) of Das-Dennis points and $\dot{\mathbf{r}}^{(k)}$, as follows:

$$\mathbf{z}^{(j,k)} = \bar{\mathbf{h}}^{(j)} + \left(\dot{\mathbf{r}}^{(k)} - \mathbf{g}\right), \tag{5}$$

where $g_i = \frac{1}{H}\sum_{j=1}^{H} h_i^{(j)}$ is the centroid of the shrunken Das and Dennis points.

After the above operation, all the Das and Dennis points will lie on the unit simplex centered around the projected point $\dot{\mathbf{r}}^k$. The above procedure is repeated for all K supplied aspiration points one by one. Thus, there will be a total of $K{*}H$ reference points, stored in $\mathbf{Z}^a = [\mathbf{z}^{j,k}], j = 1, \ldots, H, \; k = 1, \ldots, K$. Now, we append M extreme points represented by an identity matrix to the set $\mathbf{Z}^a$, as follows:

$$\mathbf{Z}^a = [\mathbf{Z}^a; \mathbf{I}_{M \times M}].$$

This makes the size of $\mathbf{Z}^a$ to $(K \cdot H + M) \times M$, or a set with $K' = (K \cdot H + M)$ reference points. The extreme points are added to create extreme Pareto optimal points so the subsequent normalization process will work well. The NSGA-III algorithm uses these points as supplied reference points, $\mathbf{Z}^a$. Note that there are no additional structured reference points. The above procedure needs to be applied at every generation, as the normalization factors $(f_i^{\min}, f_i^{\max})$ change in every generation. At the end of the NSGA-III run, we consider only the single closest solution for each reference direction generated by the original reference points, except the ones corresponding to the extreme reference directions.

Figure 1 illustrates the above procedure for two supplied aspiration points $\bar{\mathbf{r}}^{(k)}$ and $\bar{\mathbf{r}}^{(k')}$. The points are projected towards the origin (indicating the ideal point of the problem). The intersection of the line with the unit hyperplane are the projected points $\dot{\mathbf{r}}^{(k)}$ and $\dot{\mathbf{r}}^{(k')}$, respectively. The shrunken Das and Dennis points, h_j, are then moved on the hyperplane through their centroid, g, translated to the projected points $\dot{\mathbf{r}}^{(k)}$ and $\dot{\mathbf{r}}^{(k')}$. M extreme points and these H shrunk points together constitute the entire set of points, Z^a, on the unit hyperplane for an NSGA-III run. Thus, R-NSGA-III requires a population of size of

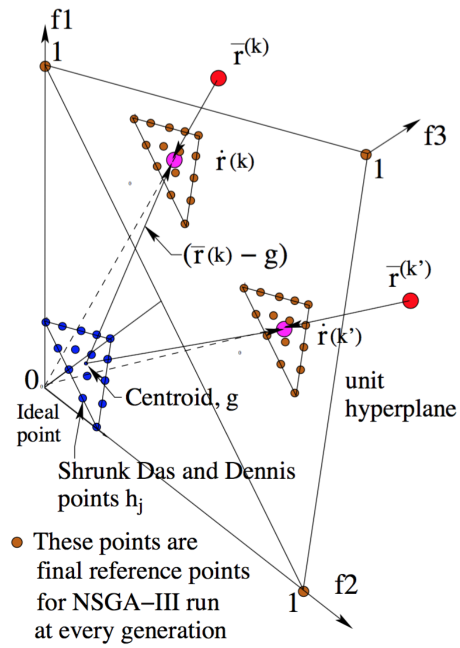


Fig. 1. A sketch of the R-NSGA-III's reference point Z^a computation procedure.

$$N_{\mathrm{III}} = \left\langle \left(M + K\binom{M+p-1}{p} \right), 2 \right\rangle, \tag{6}$$

where $\langle \alpha, 2 \rangle$ means smallest integer greater than α and is divisible by 2.

VII. Results

In this section, we present the results obtained by the above-mentioned algorithms on a number of test problems and on an engineering design problem.

First, we discuss the parameter values used in the study. For three and five objectives, the shrink factor $\mu = 0.1$ and 0.05 are used, respectively. A reduced μ value is chosen to obtain close solutions in increasing dimension spaces. For R-NSGA-II and BR-NSGA-II, we use a population size

$$N_{\mathrm{II}} = s(K + M), \tag{7}$$

where s is the number of expected points for each aspiration point, K is the number of aspiration points, and M is the number of objectives. For DTLZ2 problems we choose $\epsilon = 0.02$ and for DTLZ4 problems $\epsilon = 0.01$, except in three-objective cases, we have used 0.02. For both methods, the maximum number of generations of 500 is used. In all problems, we use the SBX operator with probability of 0.9 and an index of 10, and a polynomial mutation [4] with $1/n$ probability and an index value of 20. When comparing two algorithms, an identical number and location of reference points are used.

1) Problem DTLZ2: First, we apply all three methods – original R-NSGA-II, proposed BR-NSGA-II, and proposed R-NSGA-III – to three-objective DTLZ2 problem. Two widely separated aspiration points are chosen: $\mathbf{z}^{(1)} = (0.4, 0.1, 0.6)^T$ and $\mathbf{z}^{(2)} = (0.8, 0.5, 0.8)^T$. R-NSGA-II and BR-NSGA-II methods need an ϵ value to find a diverse set of Pareto-optimal solutions; we use $\epsilon = 0.02$. R-NSGA-III requires a shrink factor to make a dense set of reference points; we use $\mu = 0.1$. These numbers are arbitrarily chosen, but need to be coordinated to get a similar distribution, but in this proof-of-principle study, we use these numbers to simply demonstrate the working of the proposed algorithms.

For R-NSGA-II and BR-NSGA-II, we use a population of size $N = 10(2 + 3)$ or 50. For R-NSGA-III, we use $p = 5$, so that for each aspiration point, there are $\binom{H=M+p-1}{p}$ or $\binom{7}{5}$ or 21 reference points chosen. For two reference points and three objectives, the population size using Equation 6 becomes $N_{\text{III}} = \langle(3 + 2 \times 21), 2\rangle$ or 46.

Figure 2 shows obtained trade-off points using all three methods. They all are able to find points close to the supplied aspiration points. The distribution of points by R-NSGA-III is more structured due to the use two sets of Das and Dennis points as reference points on the unit simplex close to the projected aspiration points. However, the distribution obtained by both R-NSGA-II methods are not structured.

Since R-NSGA-II and BR-NSGA-II provide similar performance, for the rest of the problems, we use our proposed BR-NSGA-III and compare with R-NSGA-III method. Next, five objective DTLZ2 problems are solved with two aspiration points. Population sizes of 76 is used for five objectives. Figure 3 presents the results obtained in parallel coordinate plots (PCPs). Two reference points $\mathbf{z}^{(1)} = (0.2, 0.2, 0.2, 0.2, 0.8)^T$ and $\mathbf{z}^{(2)} = (0.5, 0.5, 0.5, 0.5, 0.5)^T$ are used. Since the whole Pareto-optimal front for DTLZ2 problem lie on the hyper-sphere $\sum_{i=1}^{M} f_i^2 = 1$, the closest Pareto-optimal objective vector (or efficient point) can be estimated for each of these aspiration points. For the first aspiration point, the objective vector $\mathbf{f} = (0.22, 0.22, 0.22, 0.22, 0.88)^T$ and for the second aspiration point, the objective vector $\mathbf{f} = \frac{1}{\sqrt{5}}(1, \ldots, 1)^T = 0.45(1, \ldots, 1)^T$ are our target efficient points. Both plots show that solutions close to these estimated objective vectors are found. The mismatch in the diversity of solutions between two methods occurs due to the arbitrary choice of ϵ and μ values. But the results demonstrate that the proposed BR-NSGA-II and R-NSGA-III are able to locate true preferred efficient points for five-objective DTLZ2 problem.

Both methods are able to find solutions close to the estimated efficient point, instead of finding the entire Pareto-optimal front.

2) Problem DTLZ3: Here we solve a 14-variable, five-objective DTLZ3 problem with one reference point $\mathbf{z}^{(1)} = (0.1, 0.5, 0.3, 0.7, 0.1)^T$ using the R-NSGA-III procedure. We set $p = 5$ resulting in a population size of 76. DTLZ3 is a more challenging problem than DTLZ2, Figure 4 shows pareto-optimal solutions found close to the provided reference point using the R-NSGA-III method.

3) Problem DTLZ4: DTLZ4 problem is more difficult to solve than DTLZ2 problem, as it has biased density of solutions on one part of the Pareto-optimal front. Figure 5 shows the obtained preferred Pareto-optimal solutions using BR-NSGA-II and R-NSGA-III on three-objective DTLZ4 problem with identical parameter values as in DTLZ2 problem. The two aspiration points are $\mathbf{z}^{(1)} = (0.3, 0.2, 0.6)^T$ and $\mathbf{z}^{(2)} = (1, 0.5, 0.2)^T$. The estimated targeted efficient objective vectors are $\mathbf{f}^{(1)} = (0.43, 0.29, 0.86)^T$ and $\mathbf{f}^{(2)} = (0.88, 0.44, 0.18)^T$, respectively. Figure 5 indicates that points close to these target vectors are obtained.

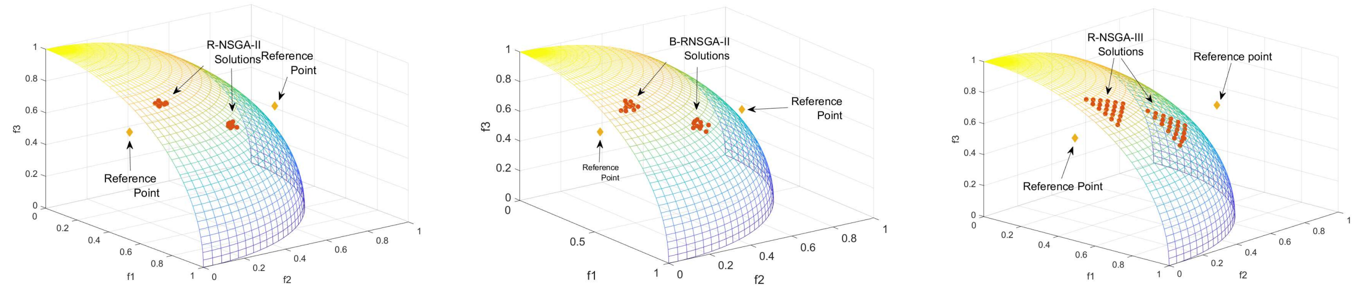
Figure 6 shows the preferred solutions obtained using R-NSGA-III on five objective DTLZ4 problem with two aspiration points $\mathbf{z}^{(1)} = (0.1, 0.2, 0.3, 0.4, 0.5)^T$ and $\mathbf{z}^{(2)} = (0.5, 0.4, 0.3, 0.2, 0.1)^T$. The respective targeted efficient objective vectors are $\mathbf{f}^{(1)} = (0.13, 0.27, 0.40, 0.54, 0.67)^T$ and $\mathbf{f}^{(2)} = (0.67, 0.54, 0.40, 0.27, 0.13)^T$ for the five-objective problem. Figures show that points close to these target objective vectors are found by R-NSGA-III. Similar results were also found using BR-NSGA-II method. In all these cases, proposed methods are able to find a concentrated set of solutions on the Pareto-optimal fronts.

4) Problem WFG5: Next, we apply both of our proposed methods to WFG5 problem. Figure 7 shows the obtained solutions with 48 population members with two aspiration points $\mathbf{z}^{(1)} = (0.8, 0.4, 3.6)^T$ and $\mathbf{z}^{(2)} = (1.6, 2, 4.8)^T$. Figures 8 shows the corresponding results on five-objective WFG5 problems with 72 population members.

5) Problem WFG6: Finally, we apply both of our proposed methods to WFG6 problem. Figure 9 shows the obtained solutions with identical parameter values as in WFG5. Figure 10 shows the corresponding results on five-objective WFG6 problem. In all these cases, focused solutions close to supplied aspiration points are found by both methods.

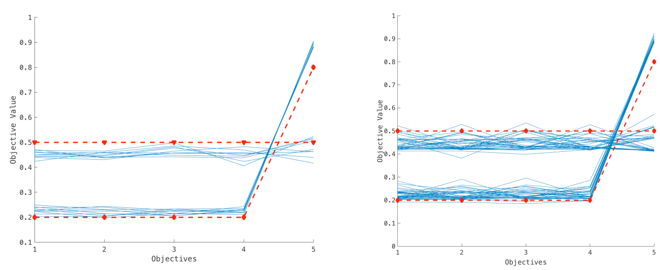
A. An Engineering Problem

Next, we consider the crashworthiness (CRASH) problem [7], which has five variables and no constraints. In order to have an idea of the Pareto-optimal front, first, we solve the problem by using the classical generative method using many achievement scalarization function (ASF) formulations [13]. There are two large gaps observed on the trade-off front, as shown in Figure 11. Thus, it becomes a natural decision-making question to establish whether gaps are truly present or they are manifestations of the optimization procedure. One way to answer the question would be to formulate a number of ASF problems by choosing reference points in the gap area of the objective space and solve respective single-objective problems. If the resulting solutions do not fall in the gap, then it indicates that there is no Pareto-optimal solution in the gap. This is what was done to start with the generative method and it seems that there is no Pareto-optimal solution in the gap. But the methods of this paper stay as other alternative methods to answer the above question, instead of resorting to a whole gamut of ASF problems solving using the classical generative approach. For this purpose, we choose two aspiration points



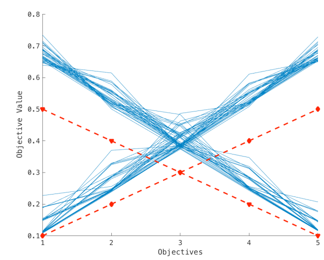
(a) R-NSGA-II. (b) BR-NSGA-II. (c) R-NSGA-III.

Fig. 2. Three-objective DTLZ2 preferred solutions identified with three methods for two reference points.



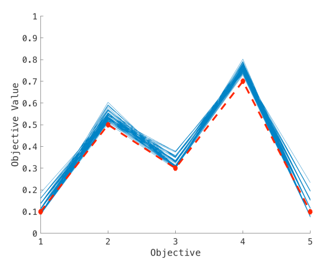
(a) BR-NSGA-II. (b) R-NSGA-III.

Fig. 3. Five-objective DTLZ2 preferred solutions (in blue) identified with R-NSGA-III for two aspiration points, shown in red.



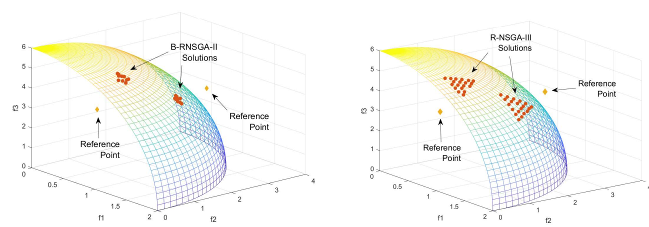
(a) 5-obj. DTLZ4 using R-NSGA-III.

Fig. 6. DTLZ4 preferred solutions found with R-NSGA-III.



(a) R-NSGA-III.

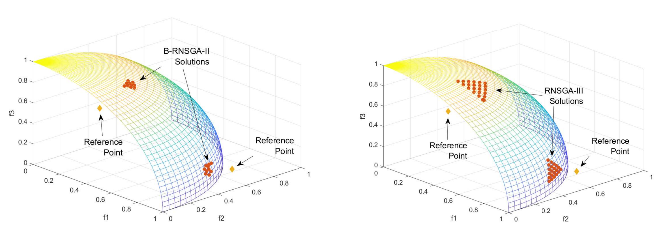
Fig. 4. DTLZ3 preferred solutions found with R-NSGA-III.



(a) BR-NSGA-II. (b) R-NSGA-III.

Fig. 7. Three-objective WFG5 preferred solutions found for two aspiration points.

in the gap area and apply both proposed BR-NSGA-II and R-NSGA-III methods with 68 population members. Figures 11 and 12 show the obtained solutions. It is clear from both plots that despite providing two aspiration points around the gaps,



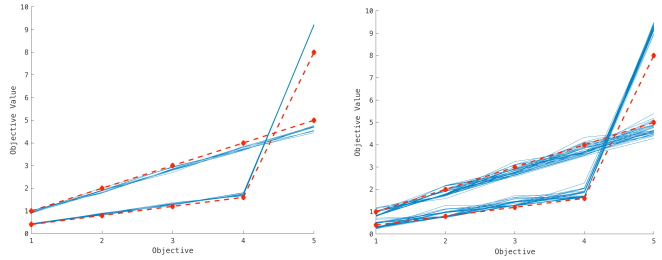
(a) BR-NSGA-II. (b) R-NSGA-III.

Fig. 5. Three-objective DTLZ4 preferred solutions found for two aspiration points.

both of our proposed methods could not find any solution in the middle of the gaps. This not only confirms the nature of trade-off front, but it also gives a DM confidence that there are actual gaps (or holes) in the Pareto-optimal front near the two aspiration points, raising interesting further analysis.

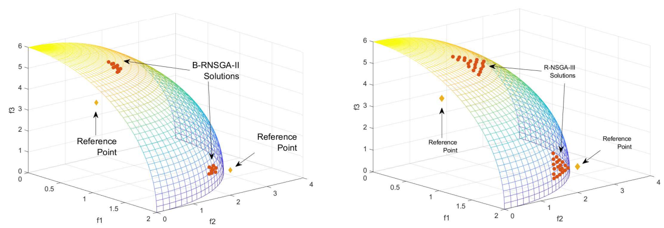
Figure 13 plots the projected points obtained by the generative method, two aspiration points and their associated reference points on the unit simplex. It can be seen that there exist a number of reference points for both aspiration points which do not apparently intersect with the the trade-off front generated by ASF method. Thus, by not finding trade-off points in the gaps in Figure 12 despite providing emphasis, R-NSGA-III supports the fact that there are gaps in the trade-off front. R-NSGA-III then ends up finding certain boundary points.

The use of an EMO method to first find an idea of the trade-off frontier and then investigating further the true nature



(a) BR-NSGA-II. (b) R-NSGA-III.

Fig. 8. Five-objective WFG5 preferred solutions found for two aspiration points.



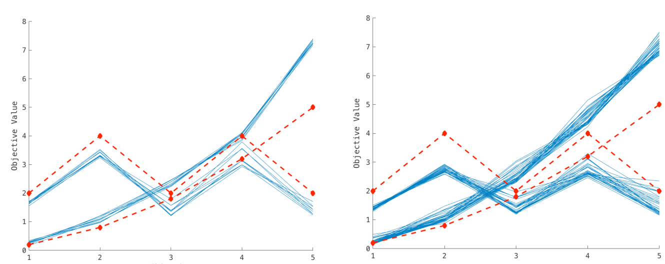
(a) BR-NSGA-II. (b) R-NSGA-III.

Fig. 9. Three-objective WFG6 preferred solutions found for two aspiration points.

of certain preferred parts of the frontier using the proposed reference point based methods will provide the decision-makers a lot of confidence in the obtained solutions.

VIII. Conclusions

Multi-objective and many-objective optimization problems must involve a decision-making task for choosing a single preferred solution. In this paper, we have extended a previously suggested reference point based NSGA-II approach to provide a more uniform emphasis in achieving solutions for all supplied aspiration points. In addition, we have extended the recently proposed NSGA-III method with the reference point concept for the same purpose. After presenting the methods in detail, we have tested them on a number of three to 5-objective test problems (DTLZ and WFG problems) with one or more aspiration points simultaneously. Finally, we have applied



(a) BR-NSGA-II. (b) R-NSGA-III.

Fig. 10. Five-objective WFG6 preferred solutions found for two aspiration points.

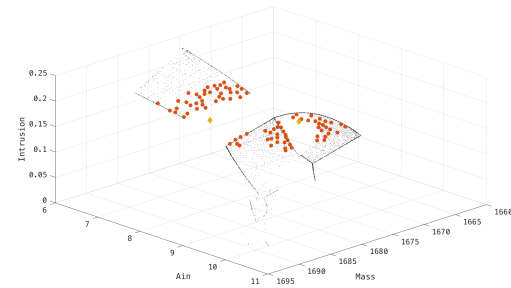


Fig. 11. BR-NSGA-II points for CRASH problem in circles with $\epsilon = 0.03$. Reference points are shown in diamond. Points obtained with generative method are in dots.

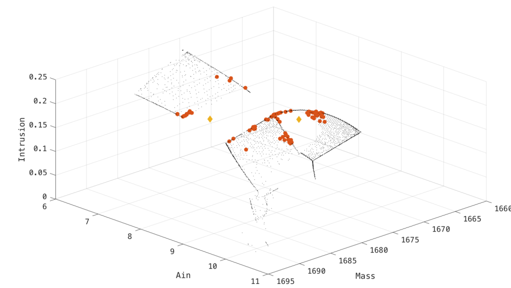


Fig. 12. R-NSGA-III points ($\mu = 0.09$) in circles for CRASH.

the proposed methods to a car crashworthiness problem to investigate if certain parts of the obtained trade-off frontier is empty or if the prior evolutionary optimization methods have failed to find any solution there.

Simulation results have worked well in achieving both tasks: (i) finding more focused trade-off solutions in the regions of interest, and (ii) validating if a certain part of the trade-off frontier has any Pareto-optimal solutions or not. The proposed methods are useful for a complete optimization-cum-decision-making purpose. If preference information is available in terms of aspiration points in the objective space, proposed methods can be applied directly. Otherwise, users can follow a two-step procedure, first finding a representative set of trade-off solutions on the entire Pareto-optimal front and then finding more focused trade-off points in the region of interest.

The study can be extended in a number of different ways. All translated Das and Dennis reference points on the unit simplex may not lead to Pareto-optimal solutions, either due to the presence of infeasible solutions or due to an absence

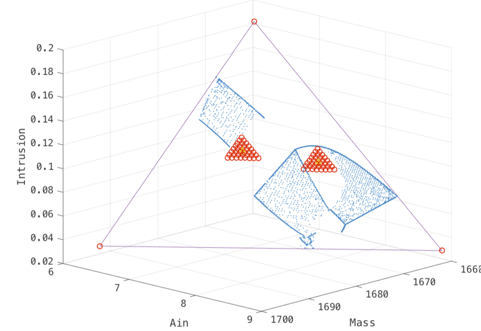


Fig. 13. Projected reference points on the unit simplex for CRASH indicate that reference directions will pass through apparent gap locations.

of any Pareto-optimal solutions there. In such cases, there is a waste of computational effort. Such non-functional reference points can be identified and relocated near to other reference points, which have already found non-dominated solutions. In this proof-of-principle study, we have chosen the shrinkage factor, μ, arbitrarily, but there is a direct relationship between the chosen μ and the obtained diversity in solutions in the objective space. R-NSGA-II and BR-NSGA-II controls the diversity of final solutions in the variable space, which may be of interest to certain design related applications. R-NSGA-III procedure can be modified to ensure a minimum distance between final solutions in the variable space. Importantly, if a certain diversity among the final solutions is desired, adaptive methods of setting the shrinkage factor, μ, need to be developed. Additionally, in this study, we have provided M extreme points on the unit simplex as reference points so that the normalization procedure works well. However, the proposed methods may find non-Pareto-optimal solutions, simply because certain critical Pareto-optimal solutions are not the target of the optimization task. It would be interesting to devise methods that would create critical Pareto-optimal points, either on the fly, or maintain from initial population, so that reference point based NSGA-II or NSGA-III methods are more likely to find Pareto-optimal solutions. The concepts developed with NSGA-II and NSGA-III can be extended to other evolutionary multi-objective and many-objective optimization algorithms.

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